MPFRCX

Multiple Precision Real and Complex Polynomial Library
Edition 0.4.2
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This manual is for MPFRCX, a library for the arithmetic of polynomials with multiple precision real or complex floating point coefficients, version 0.4.2 of May 2013.

MPFRCX Copying Conditions

1 Introduction to MPFRCX

MPFRCX is a portable library written in C for arithmetic of polynomials with arbitrary precision real or complex floating point coefficients. It is based on the GNU MP, the MPFR and the MPC libraries.

2 Installing MPFRCX

To build MPFRCX, you first have to install GNU MP, MPFR and MPC on your computer. For MPC, the minimal supported version is 1.0; the minimally required versions of GNU MP and MPFR depend on the MPC version used. You need a C compiler, preferably GCC, but any reasonable compiler should work. And you need a standard Unix 'make' program, plus some other standard Unix utility programs.

Here are the steps needed to install the library on Unix systems:

1. 'tar xzf mpfrcx-0.4.2.tar.gz'
2. 'cd mpfrcx-0.4.2'
3. './configure'

   if GMP, MPFR and MPC are installed into standard directories, that is, directories that are searched by default by the compiler and the linking tools.

   './configure --with-gmp=<gmp_install_dir>'

   is used to indicate a different location where GMP is installed.

   './configure --with-mpfr=<mpfr_install_dir>'

   is used to indicate a different location where MPFR is installed.

   './configure --with-mpc=<mpc_install_dir>'

   is used to indicate a different location where MPC is installed.

   './configure --with-gmp=<gmp_install_dir> --with-mpfr=<mpfr_install_dir> --with-mpc=<mpc_install_dir>'

   is used to indicate different locations of GMP, MPFR and MPC.

   Warning! Do not use these options if you have CPPFLAGS and/or LDFLAGS containing a -I or -L option with a directory that contains a GMP header or library file, as these options just add -I and -L options to CPPFLAGS and LDFLAGS *after* the ones that are currently declared, so that DIR will have a lower precedence. Also, this may not work if DIR is a system directory.
4. 'make'

   This compiles MPFRCX in the working directory.
5. 'make check'

   This will make sure MPFRCX was built correctly.

   If you get error messages, please report them to 'andreas.enge@inria.fr' (See Chapter 3 [Reporting Bugs], page 5, for information on what to include in useful bug reports).
6. 'make install'

   This will copy the file 'mpfrcx.h' to the directory '/usr/local/include', the file 'libmpfrcx.a' to the directory '/usr/local/lib', and the file 'mpfrcx.info' to the directory '/usr/local/share/info' (or if you passed the '--prefix' option to 'configure', using the prefix directory given as argument to '--prefix' instead of '/usr/local'). Note: you need write permissions on these directories.

2.1 Other 'make' Targets

There are some other useful make targets:

- 'mpfrcx.pdf' or 'pdf' inside the 'doc'subdirectory

  Create a PDF version of the manual, in 'mpfrcx.pdf'.

- ‘mpfrcx.html’ or ‘html’ inside the ‘doc’subdirectory

  Create an HTML version of the manual, in several pages in the directory ‘mpfrcx.html’; if you want only one output HTML file, then type ‘makeinfo --html --no-split mpfrcx.texi’ instead.
- ‘clean’

  Delete all object files and archive files, but not the configuration files.
- ‘distclean’

  Delete all files not included in the distribution.
- ‘uninstall’

  Delete all files copied by ‘make install’.

3 Reporting Bugs

If you think you have found a bug in the MPFRCX library, please investigate and report it.

There are a few things you should take into account when you compose your bug report. Please send us a minimal test case that makes it possible for us to reproduce the bug. Include instructions on how to run the test case, and why the result differs from what you would expect.

Please include compiler version information in your bug report. This can be extracted using '`cc -V`' on some machines, or, if you are using gcc, '`gcc -v`'. Also, include the output from '`uname -a`'.

Send your bug report to: '`andreas.enge@inria.fr`'.

Beware that the MPFRCX library is in a very early development stage, and some functions, while working correctly, may be terribly inefficient. You might want to send an e-mail to the above address if you are interested in one of the more exotic functions to enquire about its status.

4 MPFRCX Basics

MPFRCX provides types and functions for working with univariate polynomials, taking as coefficients either real or complex floating point numbers of arbitrary precision. The functions are collected in the library '`libmpfrcx.a`' and declared in the header file '`mpfrcx.h`'.

4.1 Nomenclature and Types

A *real polynomial* is a polynomial whose coefficients are of type `mpfr_t`. The C data type for such objects is `mpfrx_t`. All coefficients are supposed to have the same floating point precision. Besides its list of coefficients, a variable of type `mpfrx_t` contains the degree of the polynomial as an `int` and the precision of its coefficients as an `mp_prec_t`. If the degree of a polynomial increases, its list of coefficients is lengthened accordingly; on the other hand, if the degree decreases, the memory allocated to the now superfluous coefficients is not freed, unless explicitly requested by a call to `mpfrx_realloc`, see Section 5.1 [Initialisation Functions], page 7.

A *complex polynomial* is a polynomial whose coefficients are of type `mpc_t`. The C data type for such objects is `mpcx_t`. All coefficients are supposed to have the same floating point precision, and this both for their real and their imaginary parts. Otherwise, complex polynomials behave like real ones.

When calling the functions described in the following, arguments of type `mpfrx_ptr` or `mpfrx_srcptr` stand for arbitrary variables of type `mpfrx_t`; the former may be modified by the function, the latter not. The same holds for `mpcx_ptr` and `mpcx_srcptr`.

Notice that unlike for operations with real numbers of type `mpfr_t` and complex numbers of type `mpc_t`, there are no rounding modes for operations with polynomials and no precise semantics; polynomial arithmetic is realised by calls to functions on the coefficients, which may accumulate rounding errors.

4.2 Function Classes

Functions and macros working with real polynomials begin with `mpfrx_`, those treating complex polynomials begin with `mpcx_`. For the time being, there are no mixed operations.

4.3 MPFRCX Variable Conventions

As a general rule, all MPFRCX functions expect output arguments before input arguments, in analogy with GMP, MPFR and MPC.

MPFRCX allows you to use the same variable for both input and output in the same expression. For example, the main function for multiplication of real polynomials, `mpfrx_mul`, can be used like this: `mpfrx_mul (f, f, f)` to replace f by its square.

Before you can assign to an MPFRCX variable, you need to initialise it by calling one of the special initialisation functions. When you are done with a variable, you need to clear it out, using one of the functions for that purpose.

A variable should be initialised only once; after a variable has been initialised, values may be assigned to it any number of times.

5 Functions

All publicly visible functions exist with the same behaviour for real or complex polynomials; their names start with `mpfrx_` resp. `mpcx_`. For the time being, the only specific functions concern the fast Fourier transform and are internal to the library.

5.1 Initialisation Functions

An `mpfrx_t` or `mpcx_t` object must be initialised before storing the first value in it, using the function `mpfrx_init` or `mpcx_init`.

`void mpfrx_init` (*mpfrx_ptr* **`f`**, *const int* **`size`**, *const mp_prec_t* **`prec`**) [Function]
`void mpcx_init` (*mpcx_ptr* **`f`**, *const int* **`size`**, *const mp_prec_t* **`prec`**) [Function]

Initialise *f* with initial space for *size* coefficients of precision *prec*, and set it to zero. It is possible to assign a polynomial with more than *size* coefficients to *f* later on; the size of *f* is then increased automatically. Beware that *size*$= d + 1$ coefficients are needed to store a polynomial of degree *d*.

`void mpfrx_clear` (*mpfrx_ptr* **`z`**) [Function]
`void mpcx_clear` (*mpcx_ptr* **`z`**) [Function]

Free the space currently occupied by *z*. Make sure to call this function on each variable precisely once.

`void mpfrx_realloc` (*mpfrx_ptr* **`f`**, *const int* **`size`**) [Function]
`void mpcx_realloc` (*mpcx_ptr* **`f`**, *const int* **`size`**) [Function]

Changes the number of coefficients stored in *f* to *size*, which may be more or less than (or equal to) the previous size, while preserving the precision of the coefficients. If *f* still fits (that is, its degree is at most *size*-1), its value is preserved, otherwise, it is replaced by 0.

5.2 Assignment Functions

These functions assign new values to already initialised polynomials (see Section 5.1 [Initialisation Functions], page 7).

`void mpfrx_set` (*mpfrx_ptr* **`h`**, *mpfrx_srcptr* **`f`**) [Function]
`void mpcx_set` (*mpcx_ptr* **`h`**, *mpcx_srcptr* **`f`**) [Function]

Set the value of *h* from *f*. The precision of *h* is preserved, and the coefficients of *f* are rounded if the target precision is lower.

`void mpfrx_swap` (*mpfrx_ptr* **`f`**, *mpfrx_ptr* **`g`**) [Function]
`void mpcx_swap` (*mpcx_ptr* **`f`**, *mpcx_ptr* **`g`**) [Function]

Swap the contents of the variables *f* and *g*. If their coefficients do not have the same precision, precisions are swapped as well. The effect is thus not the same as obtained by several calls to `mpfrx_set` or `mpcx_set`, respectively, with an additional temporary variable, which would keep the respective precisions of *f* and *g* unchanged.

5.3 Combined Initialisation and Assignment Functions

`void mpfrx_init_set` (*mpfrx_ptr* **`h`**, *mpfrx_srcptr* **`f`**) [Function]
`void mpcx_init_set` (*mpcx_ptr* **`h`**, *mpcx_srcptr* **`f`**) [Function]

Initialise *h* with the same precision as *f* and set its value from *f*.

5.4 Access Functions

`int mpfrx_get_prec` (*mpfrx_srcptr* **f**) [Macro]
`int mpcx_get_prec` (*mpcx_srcptr* **f**) [Macro]

Return the common precision of the coefficients of *f*.

`void mpfrx_set_prec` (*mpfrx_ptr* **f**, *const mp_prec_t* **prec**) [Function]
`void mpcx_set_prec` (*mpcx_ptr* **f**, *const mp_prc_t* **prec**) [Function]

Set the precision of the coefficients of *f* to *prec* and replace *f* by the zero polynomial. The effect is similar to a call to `mpfrx_clear` resp. `mpcx_clear` followed by a call to `mpfrx_init` resp. `mpcx_init`, but the number of coefficients in the polynomial is kept.

`int mpfrx_get_deg` (*mpfrx_srcptr* **f**) [Macro]
`int mpcx_get_deg` (*mpcx_srcptr* **f**) [Macro]

Return the degree of *f*, which is -1 for the zero polynomial.

`void mpfrx_set_deg` (*mpfrx_ptr* **f**, *const int* **deg**) [Function]
`void mpcx_set_deg` (*mpcx_ptr* **f**, *const int* **deg**) [Function]

Set the degree of *f* to *deg* while preserving the coefficients. If the degree increases, the new coefficients are set to NaN and need to be set manually before computing with the variable, see `mpfrx_set_coeff` and `mpcx_set_coeff`. If necessary, new coefficients are allocated.

`mpfr_ptr mpfrx_get_coeff` (*mpfrx_srcptr* **f**, *const unsigned int* **i**) [Function]
`mpc_ptr mpcx_get_coeff` (*mpcx_srcptr* **f**, *const unsigned int* **i**) [Function]

Return a pointer to coefficient *i* of *f*, or NULL if the degree of *f* is less than *i*.

`void mpfrx_set_coeff` (*mpfrx_ptr* **f**, *const unsigned int* **i**, *mpfr_srcptr* **coeff**) [Function]
`void mpcx_set_coeff` (*mpcx_ptr* **f**, *const unsigned int* **i**, *mpc_scrptr* **coeff**) [Function]

Set the coefficient *i* of *f* to *coeff*. If the current degree of *f* is smaller than *i*, then the degree of *f* is set to *i*; intermediate coefficients are set to NaN.

5.5 Comparison and Miscellaneous Functions

`int mpfrx_cmp` (*mpfrx_srcptr* **f**, *mpfrx_srcptr* **g**) [Function]
`int mpcx_cmp` (*mpcx_srcptr* **f**, *mpcx_srcptr* **g**) [Function]

Return 0 if *f* equals *g* and -1 if not. The coefficients of *f* and *g* are compared one by one; so even if the two polynomials have different precisions, they may be recognised as equal.

`void mpfrx_urandom` (*mpfrx_ptr* **f**, *int* **deg**, *gmp_rand_state_t* **state**) [Function]
`void mpcx_urandom` (*mpcx_ptr* **f**, *int* **deg**, *gmp_rand_state_t* **state**) [Function]

If *deg*<0, set *f* to be the 0 polynomial. Otherwise, generate a uniformly distributed random degree between 0 and *deg* (inclusive), and a random polynomial of this degree. Each coefficient is chosen uniformly at random in the unit interval $[0, 1]$ resp. the unit square $[0, 1] \times [0, 1]$; except for the leading coefficient, which cannot be 0.

state is a `gmp_randstate_t` structure which should be created using the GMP `rand_init` function, see the GMP manual.

`const int MPFRCX_VERSION_MAJOR` [Macro]
`const int MPFRCX_VERSION_MINOR` [Macro]
`const int MPFRCX_VERSION_PATCHLEVEL` [Macro]

`const char* MPFRCX_VERSION_STRING` [Macro]
The major, minor and patchlevel version number of the library. These are concatenated and separated by '.' to form the version string; '-dev' is added to the version string of development snapshots.

`const char * mpfrcx_get_version` (*void*) [Function]
Return the MPFRCX version as a null-terminated string.

5.6 Input and Output Functions

The following function writes a polynomial to an output stream. When using it, you need to include '`stdio.h`' *before* '`mpfrcx.h`'.

`size_t mpfrx_out_str` (*FILE** `stream`, *int* `base`, *size_t* `n_digits`, *mpfrx_srcptr* `f`) [Function]
`size_t mpcx_out_str` (*FILE** `stream`, *int* `base`, *size_t* `n_digits`, *mpcx_srcptr* `f`) [Function]
Output *f* to *stream* using the given *base* and the given number of digits *n_digits* for the coefficients. If *n_digits* is 0, then a suitable number of digits is chosen so that reading the polynomial into a variable of the same precision as *f* yields the same polynomial again (this input function needs yet to be written...).

The output starts with an opening bracket `'('`, followed by the degree and a list of coefficients in decreasing order of degree (separated by spaces) and a closing bracket `')'`.

Return the number of written characters.

5.7 Basic Polynomial Arithmetic

`void mpfrx_add` (*mpfrx_ptr* `h`, *mpfrx_srcptr* `f`, *mpfrx_srcptr* `g`) [Function]
`void mpcx_add` (*mpcx_ptr* `h`, *mpcx_srcptr* `f`, *mpcx_srcptr* `g`) [Function]
Set *h* to $f + g$.

`void mpfrx_sub` (*mpfrx_ptr* `h`, *mpfrx_srcptr* `f`, *mpfrx_srcptr* `g`) [Function]
`void mpfrx_si_sub` (*mpfrx_ptr* `h`, *const long int* `f`, *mpfrx_srcptr* `g`) [Function]
`void mpcx_sub` (*mpcx_ptr* `h`, *mpcx_srcptr* `f`, *mpcx_srcptr* `g`) [Function]
`void mpcx_si_sub` (*mpcx_ptr* `h`, *const long int* `f`, *mpcx_srcptr* `g`) [Function]
Set *h* to $f - g$.

`void mpfrx_neg` (*mpfrx_ptr* `h`, *mpfrx_srcptr* `f`) [Function]
`void mpcx_neg` (*mpcx_ptr* `h`, *mpcx_srcptr* `f`) [Function]
Set *h* to $-f$.

`void mpfrx_mul` (*mpfrx_ptr* `h`, *mpfrx_srcptr* `f`, *mpfrx_srcptr* `g`) [Function]
`void mpcx_mul` (*mpcx_ptr* `h`, *mpcx_srcptr* `f`, *mpcx_srcptr* `g`) [Function]
Set *h* to $f * g$.

`void mpfrx_rem` (*mpfrx_ptr* `r`, *mpfrx_srcptr* `f`, *mpfrx_srcptr* `g`) [Function]
`void mpcx_rem` (*mpcx_ptr* `r`, *mpcx_srcptr* `f`, *mpcx_srcptr* `g`) [Function]
Set *r* to the remainder of *f* divided by *g*.

5.8 Higher Level Functions

`void mpfrx_root (`*mpfr_ptr* `root`, *mpfrx_srcptr* `f`) [Function]
`void mpcx_root (`*mpc_ptr* `root`, *mpcx_srcptr* `f`) [Function]

Computes a root of *f*. The variable *root* is supposed to contain an initial approximation, that is refined via Newton iterations until it does not change any more. No special care is taken to avoid infinite loops.

5.9 Tree Based Functions

The following functions implement asymptotically fast operations on arrays of polynomials, usually through the use of subproduct trees. Such a tree is a binary tree constructed from an array of polynomials by storing these polynomials in the leaves of the tree. Each parent node contains the product of the child nodes, so that the root of the tree contains the product of all the leaves. Internally, subproduct trees are variables of types `mpfrx_tree_t` and `mpcx_tree_t`. (Analogously to the situation with polynomials, in the following, `mpfrx_tree_ptr` and `mpfcx_tree_ptr` are used for variables of type `mpfrx_tree_t` and `mpcx_tree_t` that may be modified by the function, and `mpfrx_tree_srcptr` and `mpfcx_tree_srcptr` for variables that are not modified.)

`void mpfrx_tree_init (`*mpfrx_tree_ptr* `t`, *int* `no`, *mpfr_prec_t* `prec`) [Function]
`void mpcx_tree_init (`*mpcx_tree_ptr* `t`, *int* `no`, *mpfr_prec_t* `prec`) [Function]

Initialises *t* as a subproduct tree with *no* leaves in which all polynomials stored in the nodes will have precision *prec*. All nodes are initialised by a call to `mpfrx_init` or `mpcx_init`, respectively.

`void mpfrx_tree_clear (`*mpfrx_tree_ptr* `t`) [Function]
`void mpcx_tree_clear (`*mpcx_tree_ptr* `t`) [Function]

Frees the subproduct tree referenced by *t*, and all the polynomials stored in its nodes by calls to `mpfrx_clear` or `mpcx_clear`, respectively.

`void mpfrx_subproducttree (`*mpfrx_tree_ptr* `t`, *mpfrx_t* `*factors`) [Function]
`void mpcx_subproducttree (`*mpcx_tree_ptr* `t`, *mpcx_t* `*factors`) [Function]

Computes the subproduct tree *t* whose leaves contains the polynomials in the array *factors*. The variable *t* needs to have been initialised by a call to `mpfrx_tree_init` or `mpcx_tree_init`, respectively, and *factors* needs to contain at least as many elements as there are leaves in *t*. (If there are more elements in *factors*, the last ones are ignored.)

So a typical usage is a call to `mpfrx_tree_init`, followed by a call to `mpfrx_subproducttree`, and finally a call to `mpfrcx_tree_clear`.

factors is not modified by the function.

`void mpfrx_tree_get_root (`*mpfrx_ptr* `f`, *mpfrx_tree_srcptr* `t`) [Function]
`void mpcx_tree_get_root (`*mpcx_ptr* `f`, *mpcx_tree_srcptr* `t`) [Function]

Assigns the root of the tree *t* to *f*.

For instance, if *t* has been obtained by a call to `mpfrx_subproducttree` or `mpcx_subproducttree`, this retrieves the product of all polynomials on the leaves.

`void mpfrx_reconstruct (`*mpfrx_ptr* `h`, *mpfrx_t** `factors`, *int* `no`) [Function]
`void mpcx_reconstruct (`*mpcx_ptr* `h`, *mpcx_t** `factors`, *int* `no`) [Function]

Computes the product of the first *no* polynomials in the array *factors* and stores it in *h* (which may be one of the elements of *factors*; apart from that, *factors* is not modified).

The same effect could be obtained by a call to `mpfrx_subproducttree` or `mpcx_subproducttree`, respectively, followed by `mpfrx_tree_get_root` or `mpcx_tree_get_root`, respectively. But to save space by a factor of O(log(`no`)), this function uses a separate implementation.

void mpfrx_hecke (*mpfrx_ptr* **rop**, *mpfrx_tree_srcptr* **subproducts**, *mpfrx_t* ***vals**) [Function]
void mpcx_hecke (*mpcx_ptr* **rop**, *mpcx_tree_srcptr* **subproducts**, *mpcx_t* ***vals**) [Function]

Assume that *t* has been created (by a call to `mpfrx_subproducttree` or `mpcx_subproducttree`, respectively) with the elements of the array *factors* on its leaves, and let *f* be the product of all the elements in *factors* (or, equivalently, the polynomial at the root of *subproducttree*). Then the function computes *vals*[i]**f*/*factors*[i] and returns the result in *rop*. It can be used to compute the Hecke representation of algebraic numbers, whence its name.

void mpfrx_product_and_hecke (*mpfrx_t* ***rop**, *mpfrx_t* ****vals**, *int* **no_pols**, *int* **no_factors**) [Function]
void mpcx_product_and_hecke (*mpcx_t* ***rop**, *mpcx_t* ****vals**, *int* **no_pols**, *int* **no_factors**) [Function]

Combines one call to `mpfrx_subproducttree` (resp. `mpcx_subproducttree`) and one or more calls to `mpfrx_hecke` (resp. `mpcx_hecke`) into one function, which allows to not store the subproduct tree and thus to conserve memory without computational overhead. The function behaves as if a subproduct tree were created from *vals*[0], which needs to contain *no_factors* elements; the root of the tree is returned in *rop*[0]. For *i* from 1 to *no_pols-1*, it then behaves as if it called `mpfrx_hecke` (resp. `mpcx_hecke`) with this subproduct tree and *vals*[i], which needs to also contain *no_factors* values; the result of the operation is stored in *rop*[i].

void mpfrx_multieval (*mpfr_t** **values**, *mpfr_t** **args**, *int* **no**, *mpfrx_t* **f**) [Function]
void mpcx_multieval (*mpc_t** **values**, *mpc_t** **args**, *int* **no**, *mpcx_t* **f**) [Function]

Evaluates the polynomial *f* in the first *no* elements of the array *args*, and store the values in the first *no* entries of *values* (that must exist and already be initalised).

Contributors

The main developer of the MPFRCX library is Andreas Enge.

References

- Torbjörn Granlund et al. `gmp` – GNU multiprecision library. Version 5.0.2, http://gmplib.org/.
- Guillaume Hanrot, Vincent Lefèvre, Patrick Pélissier, Philippe Théveny and Paul Zimmermann. `mpfr` – A library for multiple-precision floating-point computations with exact rounding. Version 3.1.0, http://www.mpfr.org/.
- Andreas Enge, Mickaël Gastineau, Philippe Théveny and Paul Zimmermann. `mpc` – A library for multiprecision complex arithmetic with exact rounding. Version 0.9, http://mpc.multiprecision.org/.

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Version 1.3, 3 November 2008

0. PREAMBLE

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